A Universal Logic Theorem Proving Approach

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This draft document, which I have produced in a rush, is available at http://www.christoph-benzmueller.de/2016-SFO/tutorial.pdf; please forgive any typos, errors, trivialities, etc.

1 (Before the tutorial starts) Installing Isabelle

The Isabelle proof assistant is available at https://isabelle.in.tum.de/. There you also find various tutorials and documentations. The Isabelle system should ideally be installed before the tutorial starts, since this will take a while. Everything else below can be download and installed on the fly.

2 First Steps: Quantified Modal Logics

2.1 Install Convenient Abbreviations for Logic Embedding

Download the file http://www.christoph-benzmueller.de/2016-SFO/Isabelle/abbrevs and store it on your computer at `~/.isabelle/Isabelle2016/jedit/abbrevs`.

In a shell you may simply do this as follows:

```
cd ~/.isabelle/Isabelle2016/jedit/
mv abbrevs abbrevs.save.1
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/abbrevs
```

Now go to `Isabelle2016>Preferences>Abbreviations` and activate `Space bar expands abbrevs`. Then klick `Apply` and `OK`.

By this procedure you have activated the following, very convenient abbreviations for entering connectives in the embedding approach (which are displayed in Isabelle in boldface). In order to enable their display you need to hit `space` after entering the shortcuts.

shortcut	alternative latex-like input	displayed as
`mneg`	`\bol\not`	$\neg$
`mor`	`\bol\or`	$\vee$
`mand`	`\bol\and`	$\wedge$
`mimpl`	`\bol\right`	$\rightarrow$
`mequiv`	`\bol\leftr`	$\leftrightarrow$
`mall`	`\bol\for`	$\forall$
`mexi`	`\bol\exi`	$\exists$
`mnegpred`		$^\neg$
`mvalid`	`\lf\rf`	$\lfloor\rfloor$

2.2 Download the Logic Embedding of Quantified Modal Logic

Create a working directory and change the directory accordingly. Then download the Isabelle file http://www.christoph-benzmueller.de/2016-SFO/Isabelle/QML.thy

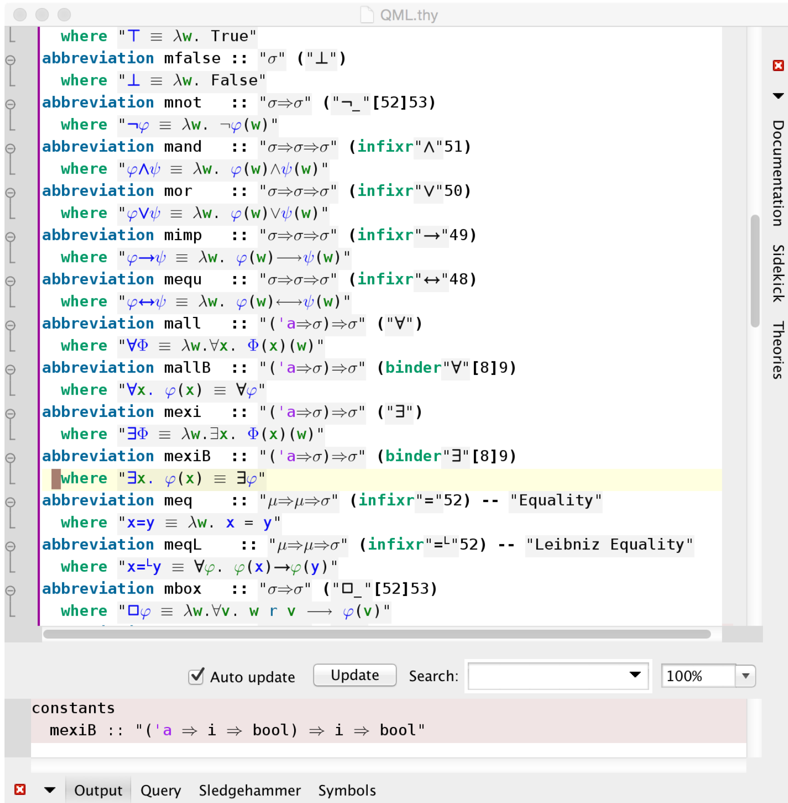

Figure 1: File QML.thy

```
mkdir ~/LogicEmbeddingsTutorial
cd ~/LogicEmbeddingsTutorial
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/QML.thy
```

You may also want to download the following example file:
http://www.christoph-benzmueller.de/2016-SFO/Isabelle/QMLex1.thy

```
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/QMLex1.thy
```

2.3 Inspect the Encoding of the Logic Embedding

Start Isabelle and open file `QML.thy` (`File>Open>QML.thy>Open`). Read and inspect the file (see Fig 1); try to understand the embedding (explanations will be given in classroom). Related information can be found here:

- Quantified Multimodal Logics in Simple Type Theory, In Logica Universalis (Special Issue on Multimodal Logics), volume 7, number 1, pp. 7-20, 2013. Available http://christoph-benzmueller.de/papers/J23.pdf.
- Slides: http://christoph-benzmueller.de/papers/2015-Tableaux.pdf http://christoph-benzmueller.de/papers/2016-Berkeley.pdf

2.4 Entering Formulas and Proving Them

In this part of the tutorial we will get familiar with theorem proving in Isabelle/HOL. Our focus will be on proof automation with tools inbuilt to Isabelle and with external reasoners. However,

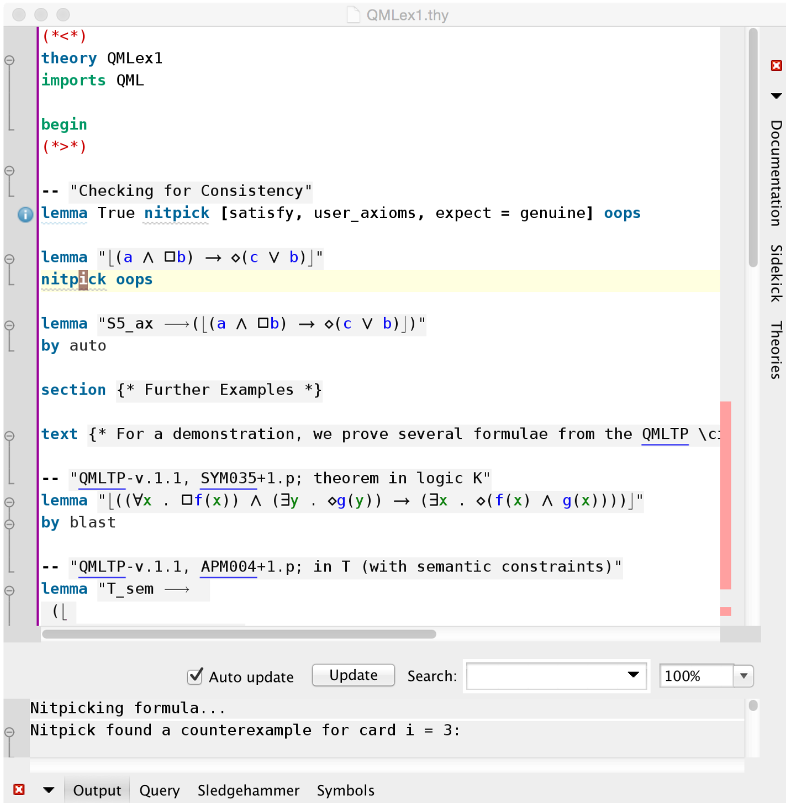

Figure 2: File QMLex1.thy

we will also briefly address interactive theorem proving in the Isar proof language. Here are some useful texts for studying:

- Hammering Away – A Users Guide to Sledgehammer for Isabelle/HOL (Jasmin Christian Blanchette), 2015. Available here http://isabelle.in.tum.de/dist/doc/sledgehammer.pdf.
- Nitpick: A Counterexample Generator for Isabelle/HOL Based on the Relational Model Finder Kodkod (Jasmin Christian Blanchette), 2013. Available **here**.
- The Isabelle/Isar Reference Manual (Makarius Wenzel), 2015.
  Available at http://isabelle.in.tum.de/doc/isar-ref.pdf.

During the tutorial we will inspect some of the examples in file `QMLex1.png` (see Fig.2).

2.4.1 Exercise

Make up some little toy examples in propositional, first-order or higher-order modal logic with pen and paper and subsequently encode them in Isabelle/HOL utilising the semantic embedding provided in file `QML.ax`. Try to prove them with Sledgehammer or disprove them with Nitpick.

2.4.2 Exercise

Encode instances of the Barcan Formula ($\forall x.\Box F(x) \longrightarrow \Box\forall x.F(x)$; domains cannot grow) and the Converse Barcan Formula ($\Box\forall x.F(x) \longrightarrow \forall x.\Box F(x)$; domains cannot shrink) and study their validity within Isabelle/HOL utilising the semantic embedding approach. Try to prove them with Sledgehammer or disprove them with Nitpick.

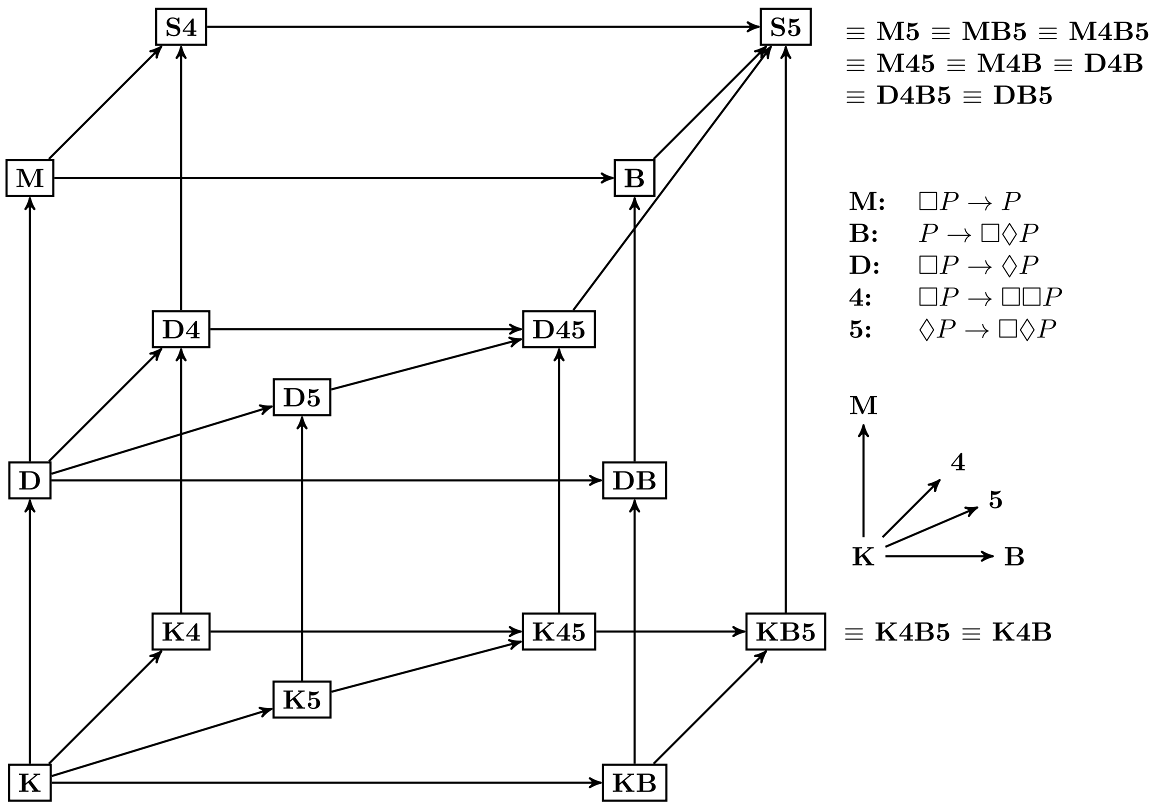

Figure 3: Modal Logic Cube

2.4.3 Exercise

Encode the propositions 3.2, 3.3, 4.2, and 5.2 from the following paper:

- Can Modalities Save Naive Set Theory? (Peter Fritz, Harvey Lederman, Tiankai Liu and Dana Scott), 2015 (draft). Available at http://logic.berkeley.edu/colloquium/ScottModalPaper.pdf

Prove them automatically using the embedding approach in Isabelle/HOL.

3 Meta-logical Reasoning

3.1 Verification of the Modal Logic Cube

In this part of the tutorial we discuss an automated verification of the well-known modal logic cube (see Fig. 3) in Isabelle/HOL, in which we prove the inclusion relations between the cubes logics using automated reasoning tools. Prior work addresses this problem but without restriction to the modal logic cube, and using encodings in first-order logic in combination with first-order automated theorem provers. In contrast, our solution is more elegant, transparent and effective. It employs the introduced embedding of quantified modal logic in classical higher-order logic. Automated reasoning tools, such as Sledgehammer with LEO-II, Satallax and CVC4, Metis and Nitpick, are employed to achieve full automation. Though successful, the experiments also motivate some technical improvements in the Isabelle/HOL tool.

Here is some further material on this work

- Systematic Verification of the Modal Logic Cube in Isabelle/HOL (Christoph Benzmüller, Maximilian Claus, Nik Sultana), In PxTP 2015, EPTCS, volume 186, pp. 27-41, 2015. Available at http://christoph-benzmueller.de/papers/C47.pdf

- Slides at: http://christoph-benzmueller.de/papers/2015-PxTP.pdf
- Isabelle source files: http://www.christoph-benzmueller.de/2016-SFO/Isabelle/mcube-final.zip

```
cd ~/LogicEmbeddingsTutorial
mkdir ModalCube
cd ModalCube
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/mcube-final.zip
unzip mcube-final.zip
cd mcube
ls
```

You will find the file `ModalCube.thy`. We will discuss the content of this file in the tutorial. Feel free to play with it!

3.2 Exercise

Prove in Isabelle that the following options to axiomatise modal logic S5 are all equivalent. S5: M5 ⇔ MB5 ⇔ M4B5 ⇔ M45 ⇔ M4B ⇔ D4B ⇔ D4B5 ⇔ DB5. Try it with the semantic and the syntactic version of the respective axioms.

3.3 Demo of the Isabelle-build Tool

We will demonstrate how LaTeX quality publications can be produced directly from Isabelle source documents.

```
cd ~/LogicEmbeddingsTutorial/ModalCube/mcube
/Applications/Isabelle2015.app/Isabelle/bin/isabelle build -D .
open ~/LogicEmbeddingsTutorial/ModalCube/mcube/doc/document.pdf
```

4 Intuitionistic Logic

4.1 Gödel Translation

We combine Gödels interpretation of propositional intuitionistic logic in propositional modal logic S4 with our above embedding in order to provide a sound and complete embedding of propositional intuitionistic logic into HOL.

Further reading:

- Multimodal and Intuitionistic Logics in Simple Type Theory (Christoph Benzmüller, Lawrence Paulson), In The Logic Journal of the IGPL, Oxford University Press, volume 18, number 6, pp. 881-892, 2010. Available at: http://christoph-benzmueller.de/papers/J21.pdf
- Eine Interpretation des Intuitionistischen Aussagenkalküls (Kurt Gödel), Ergebnisse eines Mathematischen Kolloquiums, 8:3940, 1933.

The relevant files are

```
cd ~/LogicEmbeddingsTutorial
mkdir IntuitionisticLogic
cd IntuitionisticLogic
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/Intuitionistic.thy
```

We will discuss the content of this file during the tutorial.

4.2 Exercise

Analyse the validity resp. countersatisfiability of the Law of Excluded Middle, Double Negation Elimination, the Law of Contradiction and Ex Falso Quodlibet in Isabelle.

4.2.1 Exercise

Make up your own examples in propositional intuitionistic logic and try to prove or refute them in Isabelle.

4.2.2 Exercise

Remove axiom S4 in file `Intuitionistic.thy` and identify and report the changes regarding the investigated principles in that file.

4.2.3 Exercise

Add further connectives and quantifiers to the file `Intuitionistic.thy`, so that we obtain a corresponding embedding of quantified intuitionistic logic in HOL.

4.2.4 Exercise

Make up your own examples in quantified intuitionistic logic and try to prove or refute them in Isabelle.

4.3 McKinsey-Tarski Translation

A slightly different mapping of propositional intuitionistic logic into modal logic S4 has been proposed by McKinsey and Tarski.

- McKinsey, J. C. C.; Tarski, Alfred. Some Theorems About the Sentential Calculi of Lewis and Heyting. J. Symbolic Logic 13 (1948), no. 1, 1–15.

You find this embedding in the second half of file `Intuitionistic.thy`.

4.3.1 Exercise

Analyse the validity resp. countersatisfiability of the Law of Excluded Middle, Double Negation Elimination, the Law of Contradiction and Ex Falso Quodlibet in Isabelle.

4.3.2 Exercise

Analyse the equivalence of both embeddings in Isabelle.

5 The Ontological Argument

Kurt Gödel's ontological argument for God's existence has been formalized and automated on a computer with higher-order automated theorem provers. From Gödel's premises, the computer proved: necessarily, there exists God. On the other hand, the theorem provers have also confirmed prominent criticism on Gödel's ontological argument, and they found some new results about it.

In this part of the tutorial we will analyse variants of the Ontological Argument with HOL ATPs.

Here is some further material on this work

./Images/Scott.png

Figure 4: File Scott.thy: Dana Scott's variant of Gödel's Ontological Argument

- Automating Gödel's Ontological Proof of God's Existence with Higher-order Automated Theorem Provers (Christoph Benzmüller, Bruno Woltzenlogel Paleo), In ECAI 2014 (Torsten Schaub, Gerhard Friedrich, Barry O'Sullivan, eds.), IOS Press, Frontiers in Artificial Intelligence and Applications, volume 263, pp. 93 – 98, 2014. Available at: http://christoph-benzmueller.de/papers/C40.pdf
- Invited Talk: On a (Quite) Universal Theorem Proving Approach and Its Application in Metaphysics (Christoph Benzmüller), In TABLEAUX 2015 (Hans De Nivelle, ed.), Springer, LNAI, volume 9323, pp. 209-216, 2015. Available at: http://christoph-benzmueller.de/papers/C50.pdf
- Slides from my UC Berkeley talk at: http://christoph-benzmueller.de/papers/2015-Berkeley.pdf

First we download some files to be studied to be reused in the subsections below:

```
cd ~/LogicEmbeddingsTutorial
mkdir OntologicalArgument
cd OntologicalArgument
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/QML.thy
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/Scott.thy
```

5.1 Scott's Variant

The file `Scott.thy` contains Dana Scott's version of Gödel's ontological argument (see Fig. 4). We will discuss the content in classroom.

5.1.1 Exercise

The proof in file `Scott.thy` currently uses logic S5. Try to experimentally identify the weakest modal logic in which each of the theorems is provable. Use Nitpick to eventually construct countermodels.

5.2 Inconsistency of Gödel's Axioms

I will report on the discovery and verification of the inconsistency in Gödels ontological argument with reasoning tools for higher-order logic. Despite the popularity of the argument since the appearance of Gödels manuscript in the early 70's, the inconsistency of the axioms used in the argument remained unnoticed until 2013, when it was detected automatically by the higher-order theorem prover LEO-II.

Understanding and verifying the refutation generated by LEO-II turned out to be a time-consuming task. Its completion required the reconstruction of the refutation in the Isabelle/HOL proof assistant.

The following file contains a rational reconstruction of the inconsistency argument. We will discuss this file in the tutorial.

```
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/Inconsistency_K.thy
```

5.3 Actualist versus Possibilist Quantification

We will study whether changing from constant domain semantics (possibilist quantification) to varying domain semantics (actualist quantification) changes the validity of Scott's version of the ontological argument. First download the following files:

```
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/QMLvar.thy
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/Scottvar.thy
```

Then inspect the file `QMLvar.thy` which in addition to what had before now contains the additional quantifiers $\forall^i$ and $\exists^i$ (they are restricted to individuals only). These quantifiers realise varying domain semantics as opposed to constant domain semantics. The old quantifiers remain available as well.

5.4 Exercise

At the end of file `QMLvar.thy` encode instances of the Barcan Formula ($\forall x.\Box F(x) \longrightarrow \Box\forall x.F(x)$; domains cannot grow) and the Converse Barcan Formula ($\Box\forall x.F(x) \longrightarrow \forall x.\Box F(x)$; domains cannot shrink) and study their validity within Isabelle/HOL utilising the semantic embedding approach. Of course, the idea is to employ now the new quantifiers.

5.5 Exercise

Can you modify file `QMLvar.thy` in such a way that we obtain cumulative domains? Check your attempts with the Barcan Formulas.

5.6 Scott's Variant in Logic S5 with a Universal Accessibility Relation

We explore in this part a more efficient way of automating higher-order modal logic S5 with a universal accessibility relation. When assuming a universal accessibility relation the guarding clauses in the definition of the Box and the Diamond operators can be dropped.

In Scott's variants, theorem T3 (Necessarily, God exists) can now be proved in a single step from the axioms in less than 3 seconds.

The relevant files are

```
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/QML_S5U.thy
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/Scott_S5U.thy
```

6 Multimodal Logics

6.1 General Idea

The idea is to obtain multimodal logics by introducing multiple accessibility relations and by generalising the Box and Diamond operator to accept an accessibility relation as first argument. This way we can introduce indexed box operators.

More details on the embedding of multimodal logics is available here:

- Combining and Automating Classical and Non-Classical Logics in Classical Higher-Order Logic (Christoph Benzmüller), In Annals of Mathematics and Artificial Intelligence (Special issue Computational logics in Multi-agent Systems (CLIMA XI)), volume 62, number 1-2, pp. 103-128, 2011. Available at http://christoph-benzmueller.de/papers/J25.pdf.

6.1.1 Exercise

Realise this idea by appropriately modifying the file `QML.thy` from before.

6.2 Exercise

Try to prove the quantified multimodal problems 4–12 from the above article in Isabelle/HOL using your extended embedding.

7 Quantified Conditional Logic

A notion of quantified conditional logics is studied that includes quantification over individual and propositional variables. The former is supported with respect to constant and variable domain semantics. A sound and complete embedding of this framework in classical higher-order logic is presented. Using prominent examples from the literature it is demonstrated how this embedding enables effective automation of reasoning within (object-level) and about (meta-level) quantified conditional logics with off-the-shelf higher-order theorem provers and model finders.

Further reading

- Automating Quantified Conditional Logics in HOL (Christoph Benzmller), In 23rd International Joint Conference on Artificial Intelligence (IJCAI-13) (Francesca Rossi, ed.), pp. 746-753, 2013. http://christoph-benzmueller.de/papers/C37.pdf
- Cut-Elimination for Quantified Conditional Logic (Christoph Benzmller), In Journal of Philosophical Logic, 2016. (Accepted for publication, forthcoming)

  http://christoph-benzmueller.de/papers/J31.pdf

7.0.1 Exercise

Encode the embedding of quantified conditional logic from the above papers in Isabelle/HOL. You may adapt the file `QML.thy`.

The relevant files are

```
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/ConditionalLogic.thy
```

7.0.2 Exercise

Encode the Examples 1-5 from the IJCAI paper and prove or refute them with Sledgehammer or Nitpick.

8 Multivalued Logics

Classical logics are based on the bivalence principle, that is, the set of truth-values V has cardinality $|V| = 2$, usually with $V = \{T, F\}$ where T and F stand for truthhood and falsity, respectively. Many-valued logics (MVL) generalize this requirement and allow V to be a more or less arbitrary set of truth-values, often referred to as *truth-degrees*. Popular examples of many-valued logics are fuzzy logics with an uncountable set of truth-degrees, Gödel logics and Łukasiewicz logics with denumerable sets of truth-degrees, and, from the class of finitely-many-valued logics, Dunn/Belnap's four-valued logic.

Here we study an approach for automating multivalued logics based on a sixteen-valued lattice, denoted *SIXTEEN*. This system has been developed by Shramko and Wansing as a generalization of the mentioned Dunn/Belnap four-valued system to knowledge bases in computer networks and was subsequently further investigated in various contexts.

Further reading:

- Sweet SIXTEEN: Automation via Embedding into Classical Higher-Order Logic (Alexander Steen, Christoph Benzm´uller), In 7th International Conference Non-Classical Logic – Theory and Applications, Toru, Poland, 2015. Available at: http://christoph-benzmueller.de/papers/C49.pdf
- Long paper version (currently submitted) available http://christoph-benzmueller.de/papers/2016-Sixteen.pdf
- Slides available at http://christoph-benzmueller.de/papers/2016-Torun.pdf

The relevant files are

```
cd ~/LogicEmbeddingsTutorial
mkdir MultivaluedLogics
cd MultivaluedLogics
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/SIXTEEN.thy
```

8.0.1 Exercise

9 Free Logic (Scott)

Reading:

- Existence and description in formal logic (Dana Scott), 1967. Available here: **here**
- Very end of the slides from my UC Berkeley talk at: http://christoph-benzmueller.de/papers/2015-Berkeley.pdf.

The relevant files are

```
cd ~/LogicEmbeddingsTutorial
```

```
mkdir FreeLogic
cd FreeLogic
wget http://www.christoph-benzmueller.de/2016-SFO/Isabelle/FreeHOL.thy
```

10 Nominal Logics / Hybrid Logics

Reading:

- Embedding of Quantified Higher-Order Nominal Modal Logic into Classical Higher-Order Logic (Max Wisniewski, Alexander Steen), 1st International Workshop on Automated Reasoning in Quantified Non-Classical Logics (ARQNL 2014), Vienna, Austria, Proceedings, EasyChair, EasyChair Proceedings in Computing, volume 33, pp. 59–64, 2014. Available **here**
- (Section 5.4 of) Higher-Order Modal Logics: Automation and Applications (Christoph Benzmüller, Bruno Woltzenlogel Paleo), In Reasoning Web 2015 (Adrian Paschke, Wolfgang Faber, eds.), Springer, LNCS, number 9203, pp. 32-74, 2015. Available at http://christoph-benzmueller.de/papers/C46.pdf

11 Access Control Logics

Prominent access control logics can be translated in a sound and complete way into modal logic S4. We have previously outlined how normal multimodal logics, including monomodal logics K and S4, can be embedded in simple type theory and we have demonstrated that higher-order theorem provers can automate reasoning in and about them. Here we combine these results to provide a sound (and complete) embedding of different access control logics in simple type theory.

Further reading:

- Automating Access Control Logic in Simple Type Theory with LEO-II (Christoph Benzmüller), In Emerging Challenges for Security, Privacy and Trust, 24th IFIP TC 11 International Information Security Conference, SEC 2009, Pafos, Cyprus, May 18-20, 2009. Proceedings (Dimitris Gritzalis, Javier Lpez, eds.), Springer, IFIP, volume 297, pp. 387-398, 2009. Available at http://christoph-benzmueller.de/papers/C27.pdf

12 Zalta's Theory of Abstract Objects

Reading:

- Zalta's website: https://mally.stanford.edu/theory.html
- Very end of the slides from my UC Berkeley talk at: http://christoph-benzmueller.de/papers/2015-Berkeley.pdf.

This is unpublished material; please contact me for further details.

For this theory some further convenient abbreviations are available (they are already defined in file `abbrevs`): mexe, mexe1, mexe2, mexe3, menc, mtop, mbot, mthat, mlam0, mlam1, mlam, mlam2, mlam3, xt, xp, xf, xe

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<tr><td>mand</td><td>\bol\and</td><td>$\wedge$</td></tr>
<tr><td>mimpl</td><td>\bol\right</td><td>$\rightarrow$</td></tr>
<tr><td>mequiv</td><td>\bol\leftr</td><td>$\leftrightarrow$</td></tr>
<tr><td>mall</td><td>\bol\for</td><td>$\forall$</td></tr>
<tr><td>mexi</td><td>\bol\exi</td><td>$\exists$</td></tr>
<tr><td>mnegpred</td><td></td><td>$^{\neg}$</td></tr>
<tr><td>mvalid</td><td>\lf\rf</td><td>$\lfloor\,\rfloor$</td></tr>
<tr><td>mexe</td><td>(|,|)</td><td>$\llparenthesis,\rrparenthesis$</td></tr>
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